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# On a regular convex solver potential for an elastic-damage constitutive model: a theoretical analysis

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**Abstract**

This work is related with the proposition of a so-called regular or convex solver potential to be used in numerical simulations involving a certain class of constitutive elastic-damage models. All the mathematical aspects involved are based on convex analysis, which is employed aiming a consistent variational formulation of the potential and its conjugate one. It is shown that the constitutive relations for the class of damage models here considered can be derived from the solver potentials by means of sub-differentials sets. The optimality conditions of the resulting minimisation problem represent in particular a linear complementarity problem. Finally, a simple example is present in order to illustrate the possible integration errors that can be generated when finite step analysis is performed.


*Keywords:* Damage mechanics; Variational formulation; Convex analysis

## 1. Introduction

Variational methods represent a powerful tool for consistent constitutive modelling on non-linear mechanics. Specially in the elasto-plasticity field those methods have been successfully employed (Feijóo and Zouain, 1990; Comi et al., 1992; Han and Chen, 1986).

On the other hand, continuum damage mechanics has been applied aiming to model the non-linear response induced on several materials by the evolution of a microcraking process (Krajcinovic and Fonseka, 1981; Lemaitre and Chaboche, 1990).

At the São Carlos School of Engineering the issue of variational formulation of constitutive models for continuous damage materials was preliminarily addressed in 1988 (Proença, 1988). More recently, a doctoral thesis related to the subject was concluded (Balbo, 1998), where several mathematical aspects of interest were discussed and detailed, including a material instability analysis. Some results obtained in it

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have been already published and this article is in agreement with the results found in Proença and Balbo (1997a,b, 1998).

If one considers an ideal continuous elastic-damage material whose constitutive response is characterised by an initial linear elastic regime followed by a linear softening patch, the associated potential results non-convex, since it is formed by the addition of a convex part related to the elastic response plus a concave part correspondent to the linear softening regime. The non-convexity intrinsic to the problem induces numerical crises if algorithms based on the minimisation of a convex potential are employed when attempting to account for such constitutive models.

In order to avoid numerical inconsistencies when using those kinds of minimisation algorithms, this work proposes a solver convex potential, which allows verifying the elastic-damage constitutive relations in a step by step procedure.

The detailed variational formulation of the solver potential and its conjugate here presented is based on Convex Analysis concepts (Ekeland and Temam, 1976; Rockafellar, 1970; Panagiotopoulos, 1985, 1993; Mistakidis and Stavroulakis, 1998). One justify the *solver* and *convex* adopted denominations mainly by the reason that the potential is defined by an additive composition of a strictly convex elastic potential and a damage potential which is convex on the damage variable. In particular, the *solver* label is also justified by the fact that the optimality conditions correspondent to the minimisation problem of the proposed convex potential account for the linear softening branch of the constitutive model.

The paper is organized as follows. In Section 2 the local form of the constitutive relation for an ideal elastic-damage material is presented. The assumed correspondence between the damage evolution and a certain amount of energy released by the medium is one important feature of the model pointed out in this section. The amount of energy liberated for stable damage evolution is supposed to be upper bounded. Section 3 deals with the variational formulation in rates of the constitutive model. In Section 4 the solver convex potential is presented. All the propositions related to its assumed mathematical properties are detailed. In Section 5 the existence of the conjugate (dual) potential is proved. A proper form of the potential to conduce numerical simulations is then introduced in Section 6. This form involves both finite increments of strains and damage multiplier. In Section 7 a simple numerical example consisting of a truss bar submitted to axial traction is presented. The main aim is to illustrate the exact representation of the constitutive model imposed by the solver potential when a finite step analysis is performed. As a consequence of the incremental procedure eventually adopted—purely explicit, for instance—the example shows that near a fully damage state a negative residual material rigidity can be generated if the strain step overpasses the strain limit corresponding to the prescribed maximum damage work. In order to avoid such possibility a procedure to correct the strain step is then suggested. Finally, in Section 8 an extension of the potential to include the non-associative damage rule is commented.

## 2. An elastic-damage constitutive relation in rates

In what follows it is assumed that the continuous solid occupies a region $B$ in the Euclidean pointwise space (Hilbert Space with finite dimension), being $\Gamma_u$ and $\Gamma_s$ its complementary boundaries where Dirichlet and Neumann conditions are respectively prescribed. The deformation response is supposed to be enclosed on a regime of small strains. The idealised constitutive behaviour of the medium is characterised by a linear elastic domain followed by a linear softening one. Locally, the damage penalises the initial rigidity and no permanent strains remain on unloading. The elastic rigidity $E$ of the damaged medium is supposed to be a function of an amount $w$ of energy, or damage work, dissipated in correspondence to damage progress. The damage work is assumed to be upper bounded and the correspondent maximum damage level meaning a local rupture.

Actually, the assumption of no permanent strains remaining on unloading is here adopted to simplify the analysis. Nevertheless, it could be representative of concrete materials in dominant traction regimes. The linear softening regime was assumed with the same purpose. Moreover, it could also be justified as a suitable idealisation for concrete.

In order to consider an evolution process, at any point $x \in B$ the local rate form of the constitutive relations can be expressed as follows:

$$\dot{\sigma} = E(w)\dot{\varepsilon} + \dot{E}(w)\varepsilon = \dot{\sigma}^{\mathrm{e}} + \dot{\sigma}^{\mathrm{d}} \tag{1}$$

$$f(\varepsilon, w) \leqslant 0 \tag{2}$$

$$w - \overline{w} \leqslant 0 \iff g(\alpha, w) = -\alpha - (w - \overline{w}) \leqslant 0 \tag{3}$$

$$\dot{\sigma}^{\mathrm{d}} = -\dot{\lambda} h(\varepsilon, w) \tag{4}$$

$$\dot{w} = -\dot{\lambda} r(\varepsilon, w) \tag{5}$$

$$f \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda} f = 0 \tag{6}$$

$$\text{if } f = 0 \text{ then } \dot{\lambda}\dot{f} = 0, \quad \dot{f} \leqslant 0 \tag{7}$$

In Eqs. (1)–(3): $\dot{\sigma}^{\mathrm{d}}$ is the relaxed stress rate tensor due to damage effects, the scalar valued function $f$ is proposed as a criteria for damage evolution and $\overline{w}$ represents a given upper bound limit for the dissipated energy. The tensor $h(\varepsilon, w) \geqslant 0$ is assumed to be normal to the boundary surface of an 'elastic-damage potential' and $r(\varepsilon, w) \leqslant 0$ is a scalar valued function which records the previous irreversible history by means of $w$. In the associative case the tensor $h$ can be assumed as $f_\varepsilon$, representing the gradient of the scalar function $f$ with respect to the strain tensor $\varepsilon$. The complementary and consistency conditions, Eqs. (6) and (7) account for the irreversibility of the damage evolution process, respectively.

Looking at Eq. (3) a scalar *slack* variable $\alpha \geqslant 0$ is introduced, meaning the quantity of energy to be locally dissipated until rupture. Then, if one considers a scalar valued function $g(\alpha, w)$ defined as indicated in Eq. (3), an additional complementary condition may be stated:

$$g\alpha = 0 \quad \text{with } g \leqslant 0 \text{ and } \alpha \geqslant 0 \tag{8}$$

As $\dot{g} = -\dot{\alpha} - \dot{w}$ and as $\dot{g} \leqslant 0$, then $-\dot{w} \leqslant \dot{\alpha} \leqslant 0$. In particular, if $\dot{g} = 0$ then: $\dot{\alpha} = -\dot{w}$. This kind of complementarity condition will be explored later in the variational formulation.

One way to explicit the 'damage rate multiplier' $\dot{\lambda}$ follows from the consistency condition, Eq. (7), by considering Eqs. (2) and (5):

$$\dot{f} = f_\varepsilon \cdot \dot{\varepsilon} + f_w \dot{w} = f_\varepsilon \cdot \dot{\varepsilon} - \dot{\lambda} f_w r(\varepsilon, w) \tag{9}$$

Thus,

$$\dot{\lambda} = \frac{f_\varepsilon \cdot \dot{\varepsilon}}{f_w r(\varepsilon, w)} = \frac{f_\varepsilon \cdot \dot{\varepsilon}}{G} > 0 \tag{10}$$

where $G = f_w r(\varepsilon, w) > 0$ is the elastic-damage modulus, assumed to be positive. Furthermore, $f_\varepsilon . \dot{\varepsilon} > 0$ indicates that if damage evolution occurs then the deformation rate 'vector' appoints to the outside of the elastic domain $f \leqslant 0$.

By substituting Eq. (10) into Eq. (4) the expression for $\dot{\sigma}^{\mathrm{d}}$ reads:

$$\dot{\sigma}^{\mathrm{d}} = -\left(\frac{h \otimes f_\varepsilon}{G}\right)\dot{\varepsilon} \tag{11}$$

As can be observed, this tensor rate is not symmetric. Nevertheless, its symmetry can be recovered by assuming an associative rule between $h$ and $f_\varepsilon$. In such a case:

$$\dot{\sigma}^{\mathrm{d}} = -\left(\frac{f_\varepsilon \otimes f_\varepsilon}{G}\right)\dot{\varepsilon} \tag{12}$$

By combining then Eqs. (1) and (11) and considering the non-associative case the constitutive relation can be expressed as:

$$\dot{\sigma} = \left[E(w) - \left(\frac{h \otimes f_\varepsilon}{G}\right)\right]\dot{\varepsilon} \quad \text{if } \dot{\lambda} > 0 \tag{13}$$

or in the associative case as:

$$\dot{\sigma} = \left[E(w) - \left(\frac{f_\varepsilon \otimes f_\varepsilon}{G}\right)\right]\dot{\varepsilon} \quad \text{if } \dot{\lambda} > 0 \tag{14}$$

Using Eqs. (5) and (10), the damage work evolution can be expressed as:

$$\dot{w} = -\frac{f_\varepsilon \cdot \dot{\varepsilon}}{G} r \quad \text{or} \quad \dot{w} = -\frac{f_\varepsilon \cdot .\dot{\varepsilon}}{f_w} \tag{15}$$

On what concerns to the slack variable, being $\dot{g} = 0$, from Eq. (8) one obtains:

$$\dot{\alpha} = -\dot{w} = \frac{f_\varepsilon \cdot \dot{\varepsilon}}{f_w}.$$

Finally, it is possible to find a relation between $\dot{\alpha}$ and $\dot{\lambda}$, expressed by

$$\dot{\lambda} = \psi\dot{\alpha} \quad \text{with } \psi = r^{-1} \leqslant 0 \tag{16}$$

where $\psi$ is given by:

$$\psi = \frac{(f_\varepsilon \otimes \dot{\varepsilon}) \cdot E_w}{\|f_\varepsilon\|^2} \tag{17}$$

accounting for the associative case.

## 3. Variational formulation of the constitutive model in rates

In what follows some general assumptions are considered:

(i) Let $W^*$ and $W$ not empty dual associated vectorial sub-spaces to $B$, containing respectively the tensor rates of stresses and strains. The linear vector space associated to $B$ and the sub-spaces $W^*$ and $W$ are endowed with the following norms:

$$\|x_B\| = \left(\int_B x \cdot x \,\mathrm{d}B\right)^{1/2}; \quad \|\dot{\varepsilon}\|_w = \left(\int_B \dot{\varepsilon} \cdot \dot{\varepsilon}\,\mathrm{d}B\right)^{1/2}; \quad \|\dot{\sigma}\|_w = \left(\int_B \dot{\sigma} \cdot \dot{\sigma}\,\mathrm{d}B\right)^{1/2} \tag{18}$$

with $x \in B$, $\dot{\varepsilon} \in W$ and $\dot{\sigma} \in W^*$.

Between the dual spaces a so called duality product is introduced and defined as:

$$\langle \dot{\sigma}, \dot{\varepsilon} \rangle_{W \times W^*} = \int_B \dot{\sigma} \cdot \dot{\varepsilon} \, \mathrm{d}B \tag{19}$$

(ii) In the consistency and complementary Eqs. (6) and (7), $f$ and $\dot{\lambda}$ are being assumed as scalars. Although, the cases where $\dot{\lambda}$ and $f$ are assumed to be vectors could also be included. Then, for generality the damage rate multiplier $\dot{\lambda}$ and the damage criteria $f$ are here considered to belong to the dual spaces $\Lambda$ and $\Lambda^*$ in $B$, both respectively endowed with the following norms:

$$\|\dot{\lambda}\|_\Lambda = \left( \int_B \dot{\lambda} \cdot \dot{\lambda} \, \mathrm{d}B \right)^{1/2}; \quad \|f\|_{\Lambda^*} = \left( \int_B f \cdot f \, \mathrm{d}B \right)^{1/2} \tag{20}$$

Between the dual spaces $\Lambda$ and $\Lambda^*$, a duality product is also introduced and defined as:

$$\langle f, \dot{\lambda} \rangle_{\Lambda \times \Lambda^*} = \int_B f \cdot \dot{\lambda} \, \mathrm{d}B \tag{21}$$

Furthermore, it is convenient to define the following sets:

$$\Lambda_f^+ = \{ \dot{\lambda} \geqslant 0 / f \dot{\lambda} = 0 \ \ \forall x \in B \} \tag{22}$$

$$\Lambda_f = \{ \dot{\lambda} \geqslant 0, \ \ \forall x \in B \} \tag{23}$$

$$\Lambda_g = \{ \dot{\alpha} \leqslant 0, \ \ \forall x \in B \} \tag{24}$$

It must be noted that, by convenience, in Eq. (22) $f$ and $\dot{\lambda}$ were again treated as scalars.

(iii) $f = f(\varepsilon, w)$ is a *regular* (non-strictly) *convex* scalar valued function of the field $\varepsilon \in W$ and the scalar $w$; $f_\varepsilon = f_\varepsilon(\varepsilon, \lambda)$ is a linear operator of $W \times \Lambda$ in $W^* \times \Lambda$, assumed as:

(iii.1) lower and upper bounded in $\Lambda_f$, i.e., there are constants $h_0 > 0$ and $h'_0 > 0$ such that

$$h_0 \|\dot{\lambda}\|_\Lambda \geqslant \|f_\varepsilon \dot{\lambda}\|_{W^*} \geqslant h'_0 \|\dot{\lambda}\|_\Lambda \quad \forall \dot{\lambda} \in \Lambda_f \tag{25}$$

This property implies $\|\dot{\sigma}^{\mathrm{d}}\|_{W^*} = \| - f_\varepsilon \dot{\lambda}\|_{W^*} \neq 0$ and finite. Furthermore, the upper bound also implies that there is only one $\dot{\lambda} \in \Lambda_f$ such that $f_\varepsilon \dot{\lambda}$ is equal to a certain $\dot{\sigma}^{\mathrm{d}}$ prescribed.

(iii.2) $f_\varepsilon \dot{\lambda}$ is continuously dependent of $\dot{\varepsilon} \in W$, i.e., for $h_1 > 0$,

$$|\langle \dot{\varepsilon}, f_\varepsilon \dot{\lambda} \rangle| \leqslant h_1 \|\dot{\varepsilon}\|_W \|\dot{\lambda}\|_\Lambda \quad \forall \dot{\lambda} \in \Lambda, \ \forall \dot{\varepsilon} \in W \tag{26}$$

This property ensures that the rate of dissipated energy $|\langle \dot{\varepsilon}, f_\varepsilon \dot{\lambda} \rangle|$ is finite. Then, the damage process is continuous and limited.

(iv) The elastic tensor $E$ is symmetric and positive definite. Then, for $h_2 \geqslant 0$ and $h_3 \geqslant \|E\|_\infty$, where $\|E\|_\infty = \max_i \sum_{j,k,l} |E_{ijkl}|$, the following condition is valid:

$$h_2 \|\dot{\varepsilon}\|_W^2 \leqslant \langle E\dot{\varepsilon}, \dot{\varepsilon} \rangle \leqslant h_3 \|\dot{\varepsilon}\|_W^2 \quad \forall \dot{\varepsilon} \in W \tag{27}$$

(v) In the general case, the elastic-damage modulus $G$ is positive semi-definite operator such that for $h_4 \geqslant 0$ and $h_5 \geqslant \|G\|_\infty$,

$$h_4 \|\dot{\lambda}\|_\Lambda^2 \leqslant \langle G\dot{\lambda}, \dot{\lambda} \rangle \leqslant h_5 \|\dot{\lambda}\|_\Lambda^2 \quad \forall \dot{\lambda} \in \Lambda_f \tag{28}$$

In particular, the proposed elastic-damage model $G$ is a non-negative number, as defined by Eq. (10).

(vi) For $\dot{\lambda} \in \Lambda_f^+$, the complementarity and consistency conditions, Eqs. (6) and (7), respectively, may be expressed in an equivalent variational form as

$$\langle \dot{f}, \dot{\lambda}^* - \dot{\lambda} \rangle = \langle f_\varepsilon \cdot \dot{\varepsilon} - \dot{\lambda} f_w r(\varepsilon, w), \dot{\lambda}^* - \dot{\lambda} \rangle \leqslant 0, \quad \forall \dot{\lambda}^* \in \Lambda_f^+ \tag{29}$$

## 4. A solver convex potential for the elastic-damage model

Let be considered initially a potential $\phi : W \times \Lambda \to \Re$ defined as:

$$\phi(\dot{\varepsilon}, \dot{\lambda}) = \{1/2\langle E\dot{\varepsilon}, \dot{\varepsilon}\rangle + \langle -\dot{\lambda} f_{\varepsilon}, \dot{\varepsilon}\rangle + 1/2\langle G\dot{\lambda}, \dot{\lambda}\rangle\} \tag{30}$$

The idea behind such proposition is to avoid a complex treatment of the variational form of the model discussed so far, since its potential is non-convex. In order to explore only convex analysis tools, the regularizing or solver convex potential above is here introduced. The properties of the potential are discussed on what follows and warrants that the local constitutive relations, correspondent even to the elastic or elastic-damage regimes can be verified at each step of the analysis.

The proposed potential is endowed with the following properties:

(P1) $\phi$ is convex in the variables $\dot{\varepsilon} \in W$ and $\dot{\lambda} \in \Lambda_f^+$.

**Proof.** This property is ensured immediately from Eqs. (27) and (28), as the operator $E$ is symmetric, positive definite and the damage modulus $G$ is non-negative. □

(P2) $\phi$ is continuum in the variables $\dot{\varepsilon} \in W$ and $\dot{\lambda} \in \Lambda_f^+$.

**Proof.** In order to prove such proposition as a first step the continuous dependence of $\phi$ on the variable $\dot{\varepsilon} \in W$ is shown.

Then, for any $\dot{\varepsilon}^*$, $\dot{\varepsilon} \in W$, such that, $\|\dot{\varepsilon}^* - \dot{\varepsilon}\|_B\langle\delta, \delta\rangle 0$, and for any $\dot{\lambda} \in \Lambda_f^+$:

$$\begin{aligned}
|\phi(\dot{\varepsilon}^*, \dot{\lambda}) - \phi(\dot{\varepsilon}, \dot{\lambda})| &= |1/2\langle E\dot{\varepsilon}^*, \dot{\varepsilon}^*\rangle - 1/2\langle E\dot{\varepsilon}, \dot{\varepsilon}\rangle - \langle \dot{\lambda} f_{\varepsilon}, \dot{\varepsilon}^*\rangle + \langle \dot{\lambda} f_{\varepsilon}, \dot{\varepsilon}\rangle| \\
&\leqslant |1/2\langle E(\dot{\varepsilon}^* + \dot{\varepsilon}), \dot{\varepsilon}^* - \dot{\varepsilon}\rangle + \langle \dot{\lambda} f_{\varepsilon}, \dot{\varepsilon} - \dot{\varepsilon}^*\rangle| \\
&\leqslant 1/2\|E\|_{\infty}\|\dot{\varepsilon}^* + \dot{\varepsilon}\|_B\|\dot{\varepsilon}^* - \dot{\varepsilon}\|_B + \|\dot{\lambda} f_{\varepsilon}\|_{W^*}\|\dot{\varepsilon} - \dot{\varepsilon}\|_B \\
&\leqslant 1/2 h_5\|\dot{\varepsilon}^* - \dot{\varepsilon}\|_B\|\dot{\varepsilon}^* + \dot{\varepsilon}\|_B + h_0\|\dot{\lambda}\|_{\Lambda}\|\dot{\varepsilon}^* - \dot{\varepsilon}\|_B \\
&\leqslant K\|\dot{\varepsilon}^* - \dot{\varepsilon}\|_B,
\end{aligned}$$

where $K = 1/2 h_5\|\dot{\varepsilon}^* + \dot{\varepsilon}\|_B + h_0\|\dot{\lambda}\|_{\Lambda}$.

Hence, $\phi$ is continuously dependent of $\dot{\varepsilon}$.

Analogously it can be proved that $\phi$ is continuously dependent on $\dot{\lambda} \in \Lambda_f^+$. □

(P3) $\phi$ is coercive in the variables $\dot{\varepsilon} \in W$ and $\dot{\lambda} \in \Lambda_f^+$.

**Proof.** The proof of the coerciveness with respect to $\dot{\varepsilon} \in W$ for a given $\dot{\lambda} \in \Lambda_f^+$, can be constructed as follows:

$$\begin{aligned}
\phi(\dot{\varepsilon}, \dot{\lambda}) &= \{1/2\langle E\dot{\varepsilon}, \dot{\varepsilon}\rangle + \langle -\dot{\lambda} f_{\varepsilon}, \dot{\varepsilon}\rangle + 1/2\langle G\dot{\lambda}, \dot{\lambda}\rangle\} \\
&\geqslant h_2\|\dot{\varepsilon}\|_W^2 - |\langle \dot{\varepsilon}, \dot{\lambda} f_{\varepsilon}\rangle| + 1/2\langle G\dot{\lambda}, \dot{\lambda}\rangle \\
&\geqslant 1/2h_2\|\dot{\varepsilon}\|_W^2 - |\langle \dot{\varepsilon}, \dot{\lambda} f_{\varepsilon}\rangle| \\
&\geqslant 1/2 h_2\|\dot{\varepsilon}\|_W^2 - h_1\|\dot{\varepsilon}\|_W\|\dot{\lambda}\|_{\Lambda} \\
&= (1/2 h_2\|\dot{\varepsilon}\|_W - h_1\|\dot{\lambda}\|_{\Lambda})\|\dot{\varepsilon}\|_W
\end{aligned}$$

Hence, $\lim_{\|\dot{\varepsilon}\| \to +\infty} \phi(\dot{\varepsilon}, \dot{\lambda}) = +\infty$ and $\phi_1$ is coercive in $\dot{\varepsilon} \in W$.

Analogously, it can be proved that $\phi$ is coercive in $\dot{\lambda} \in \Lambda_f^+$. □

**Proposition 4.1.** *The potential $\phi$ defined by relation (30) reaches its infimum in $W$ and $\Lambda_f^+$, i.e., there is a solution for the following infimum problem,*

$$\inf_{\dot{\lambda}\in\Lambda_f^+}\inf_{\dot{\varepsilon}\in W}\phi(\dot{\varepsilon},\dot{\lambda}) = \inf_{\dot{\lambda}\in\Lambda_f^+}\inf_{\dot{\varepsilon}\in W}\{1/2\langle E\dot{\varepsilon},\dot{\varepsilon}\rangle - \langle\dot{\lambda}f_\varepsilon,\dot{\varepsilon}\rangle + 1/2\langle G\dot{\lambda},\dot{\lambda}\rangle\} \tag{31}$$

**Proof.** Being $W$ and $\Lambda_f^+$ non-empty closed sets and considering the convexity, continuity and coerciveness of $\phi$ in the variables $\dot{\varepsilon}\in W$ and $\dot{\lambda}\in\Lambda_f^+$, then according to Proposition A.2.1, Appendix A.2, $\phi$ is *weakly lower semi-continuous* (l.s.c.) and presents the *growth property*. Therefore, considering also the results ruled in the Appendix A.2 (Proposition A.2.2) $\phi$ is bounded in the variables $\dot{\varepsilon}\in W$ and $\dot{\lambda}\in\Lambda_f^+$. As a consequence of the previous results, there is an *infimum* to $\phi$ in the sets $W$ and $\Lambda_f^+$ and $\phi$ reaches its infimum in these sets, i. e., there is solution for the infimum problem given by Eq. (31).

Taking into account the present results, the former infimum problem is equivalent to:

$$\inf_{\dot{\varepsilon}\in W}\inf_{\dot{\lambda}\in\Lambda_f^+}\{1/2\langle E\dot{\varepsilon},\dot{\varepsilon}\rangle - \langle\dot{\lambda}f_\varepsilon,\dot{\varepsilon}\rangle + 1/2\langle G\dot{\lambda},\dot{\lambda}\rangle\} \qquad \square \tag{32}$$

**Remark.** The infimum form above, where the $\dot{\varepsilon}$ follows from a given $\dot{\lambda}$, is not so useful to proceed a solid mechanics analysis. Actually, a more convenient form is the one expressed by Eq. (31), where $\dot{\lambda}$ follows from a given $\dot{\varepsilon}$. Anyway, Eq. (32) will be used in the proof of the proposition that follows.

**Proposition 4.2.** *If $\phi$ admits an infimum in $W$ and $\Lambda_f^+$, then $\phi$ is l.s.c. in $W$ and $\Lambda_f^+$.*

**Proof.** Let $\dot{\varepsilon}^*\in W$ and $\dot{\lambda}^*\in\Lambda_f^+$ be assumed as the variables correspondent to the infimum of $\phi$. Then $\phi$ is l.s.c. in $\dot{\varepsilon}\in W$ and in agreement with Definition A.1.3 (Appendix A.1), and for $n\to\infty$:

$$\begin{aligned}\lim_{\dot{\varepsilon}^n\to\dot{\varepsilon}^*}\left[\inf_{\dot{\varepsilon}^n\in W}\inf_{\dot{\lambda}\in\Lambda_f^+}\phi(\dot{\varepsilon}^n,\dot{\lambda})\right] &= \lim_{\dot{\varepsilon}^n\to\dot{\varepsilon}^*}\left[\inf_{\dot{\varepsilon}\in W}\inf_{\dot{\lambda}\in\Lambda_f^+}\{1/2\langle E\dot{\varepsilon}^n,\dot{\varepsilon}^n\rangle - \langle\dot{\lambda}f_\varepsilon,\dot{\varepsilon}^n\rangle + 1/2\langle G\dot{\lambda},\dot{\lambda}\rangle\}\right]\\ &\geqslant \lim_{\dot{\varepsilon}^n\to\dot{\varepsilon}^*}\left[\inf_{\dot{\varepsilon}\in W}\{1/2\langle E\dot{\varepsilon}^n,\dot{\varepsilon}^n\rangle - \langle\dot{\lambda}^*f_\varepsilon,\dot{\varepsilon}^n\rangle + 1/2\langle G\dot{\lambda}^*,\dot{\lambda}^*\rangle\}\right]\\ &\geqslant \{1/2\langle E\dot{\varepsilon}^*,\dot{\varepsilon}^*\rangle - \langle\dot{\lambda}^*f_\varepsilon,\dot{\varepsilon}^*\rangle + 1/2\langle G\dot{\lambda}^*,\dot{\lambda}^*\rangle\} = \phi(\dot{\varepsilon}^*,\dot{\lambda}^*)\end{aligned}$$

Thus, $\lim_{\dot{\varepsilon}^n\to\dot{\varepsilon}^*}[\inf_{\dot{\varepsilon}^n\in W}\inf_{\dot{\lambda}\in\Lambda_f^+}\phi(\dot{\varepsilon}^n,\dot{\lambda})] \geqslant \phi(\dot{\varepsilon}^*,\dot{\lambda}^*)$.

Analogously it can be showed that $\phi$ is l.s.c. in $\dot{\lambda}\in\Lambda_f^+$. $\square$

**Proposition 4.3.** *The functional $\phi$ is proper in the variables $\dot{\varepsilon}\in W$ and $\dot{\lambda}\in\Lambda_f^+$.*

**Proof.** As $\phi$ is proper in the variable $\dot{\varepsilon}\in W$ then $\phi(\dot{\varepsilon},0) = 1/2\langle E\dot{\varepsilon},\dot{\varepsilon}\rangle \geqslant 0$, and it assumes a finite value in hypothesis of small displacements and strains.

Furthermore, $\phi(\dot{\varepsilon},\dot{\lambda}) > -\infty$ so, for $\dot{\varepsilon}^*\in W$ assumed as infimum for $\phi$ in $W$ and with $\dot{\lambda}\in\Lambda_f^+$:

$$\begin{aligned}\phi(\dot{\varepsilon},\dot{\lambda}) &\geqslant \{1/2\langle E\dot{\varepsilon}^*,\dot{\varepsilon}^*\rangle + \langle -\dot{\lambda}f_\varepsilon,\dot{\varepsilon}^*\rangle + 1/2\langle G\dot{\lambda},\dot{\lambda}\rangle\}\\ &\geqslant 1/2\langle E\dot{\varepsilon}^*,\dot{\varepsilon}^*\rangle + \langle -\dot{\lambda}f_\varepsilon,\dot{\varepsilon}^*\rangle \geqslant 1/2\langle E\dot{\varepsilon}^*,\dot{\varepsilon}^*\rangle - |\langle\dot{\lambda}f_\varepsilon,\dot{\varepsilon}^*\rangle| \geqslant -\infty\end{aligned}$$

Therefore, being $1/2\langle E\dot{\varepsilon}^*, \dot{\varepsilon}^*\rangle \geqslant 0$ and considering the inequality

$$-|\langle f_\varepsilon \dot{\lambda}, \dot{\varepsilon}^*\rangle| \geqslant -h_1 \|\dot{\varepsilon}^*\|_W \|\dot{\lambda}\|_\Lambda > -\infty$$

which is in agreement with Definition A.1.2 (Appendix A.1), $\phi$ is proper in the variable $\dot{\varepsilon} \in W$.

Analogously it follows that $\phi$ is proper in $\dot{\lambda} \in \Lambda_f^+$. □

**Proposition 4.4.** *$\phi$ is differentiable in the variables $\dot{\varepsilon} \in W$ and $\dot{\lambda} \in \Lambda_f^+$.*

**Proof.** By using the total differential absolute value with relation to the variables $\dot{\varepsilon} \in W$ and $\dot{\lambda} \in \Lambda_f^+$ and considering $\dot{\varepsilon}^* \in W$ and $\dot{\lambda}^* \in \Lambda_f^+$ such that $\dot{\varepsilon}^* \to \dot{\varepsilon}$ and $\dot{\lambda}^* \to \dot{\lambda}$, then:

$$\begin{aligned}
&|\phi(\dot{\varepsilon}^*, \dot{\lambda}^*) - \phi(\dot{\varepsilon}, \dot{\lambda}) - \langle E\dot{\varepsilon}, \dot{\varepsilon}^* - \dot{\varepsilon}\rangle - \langle -\dot{\lambda} f_\varepsilon, \dot{\varepsilon}^* - \dot{\varepsilon}\rangle + \langle \dot{\lambda}^* - \dot{\lambda}, f_\varepsilon \dot{\varepsilon}\rangle - \langle G\dot{\lambda}, \dot{\lambda}^* - \dot{\lambda}\rangle| \\
&\quad = |1/2\langle E(\dot{\varepsilon}^* - \dot{\varepsilon}), \dot{\varepsilon}^* - \dot{\varepsilon}\rangle + \langle -(\dot{\lambda}^* - \dot{\lambda}) f_\varepsilon, \dot{\varepsilon}^* - \dot{\varepsilon}\rangle + 1/2\langle G(\dot{\lambda}^* - \dot{\lambda}), \dot{\lambda}^* - \dot{\lambda}\rangle| \\
&\quad \leqslant 1/2 h_2 \|\dot{\varepsilon}^* - \dot{\varepsilon}\|_B^2 + h_1 \|\dot{\varepsilon}^* - \dot{\varepsilon}\|_B \|\dot{\lambda}^* - \dot{\lambda}\|_B + 1/2 h_5 \|\dot{\lambda}^* - \dot{\lambda}\|_B^2
\end{aligned}$$

Such result implies that $\phi$ is differentiable for $\dot{\varepsilon}^* \to \dot{\varepsilon}$ and $\dot{\lambda}^* \to \dot{\lambda}$.

Thus, $\phi$ is differentiable in the variables $\dot{\varepsilon} \in W$ and $\dot{\lambda} \in \Lambda_f^+$. □

**Proposition 4.5.** *If $\phi$ is convex, l.s.c., proper and differentiable in $\dot{\lambda} \in \Lambda_f^+$, then the consistency conditions of the elastic-damage model represented by the variational Eq. (29), are satisfied.*

**Proof.** Since that $\phi$ is convex, l.s.c., proper and differentiable in $\dot{\lambda}^* \in \Lambda_f^+$, then $\exists \dot{\lambda} \in \Lambda_f^+$ assumed as infimum to $\phi$ in $\Lambda_f^+$, satisfying the optimality conditions:

$$\begin{aligned}
&\langle \nabla_\lambda \phi(\dot{\varepsilon}, \dot{\lambda}^*), \dot{\lambda}^* - \dot{\lambda}\rangle \geqslant 0, \ \forall \dot{\lambda}^* \in \Lambda_f^+ \iff \langle -f_\varepsilon \cdot \dot{\varepsilon} + G\dot{\lambda}^*, \dot{\lambda}^* - \dot{\lambda}\rangle \geqslant 0, \ \forall \dot{\lambda}^* \in \Lambda_f^+ \\
&\quad \text{or} \quad \langle f_\varepsilon \cdot \dot{\varepsilon} - G\dot{\lambda}^*, \dot{\lambda}^* - \dot{\lambda}\rangle \leqslant 0, \ \forall \dot{\lambda}^* \in \Lambda_f^+
\end{aligned} \tag{33}$$

which is equivalent to consistency conditions for the elastic-damage model. □

## 5. Existence for the conjugate (dual) functional $\phi^*$

Since for some $\dot{\lambda} \in \Lambda_f^+$, $\phi$ is a convex functional, l.s.c., proper and defined in a non-empty set $W$, when written in terms of $\dot{\varepsilon} \in W$, then, in agreement with Definition A.3.1 and considering the assumption of $W^*$ to be a non-empty set, there is a conjugated potential $\phi^* : W^* \to \bar{\Re}$ verifying the following condition:

$$\phi^*(\dot{\sigma}, \dot{\lambda}) = \sup_{\dot{\varepsilon} \in W} \{\langle \dot{\sigma}, \dot{\varepsilon}\rangle - \phi(\dot{\varepsilon}, \dot{\lambda})\} \quad \forall \dot{\sigma} \in W^*. \tag{34}$$

For $\phi^*(\dot{\sigma}, \dot{\lambda}^*) < \infty$, there are sub-differential sets $\partial_\varepsilon \phi^*(\dot{\varepsilon}, \dot{\lambda}^*)$ and $\partial_\sigma \phi^*(\dot{\sigma}, \dot{\lambda}^*)$, which are closed and non-empty, such that the following relations of duality may be established:

$$\dot{\sigma} \in \partial_\varepsilon \phi(\dot{\varepsilon}, \dot{\lambda}) \iff \dot{\varepsilon} \in \partial_\sigma \phi^*(\dot{\sigma}, \dot{\lambda}) \tag{35}$$

where, being $\dot{\lambda}^* \in \Lambda_f^+$

$$\partial_\varepsilon \phi(\dot{\varepsilon}, \dot{\lambda}) = \{\gamma^* \in W^* : \forall \dot{\varepsilon}^* \in W, \phi(\dot{\varepsilon}^*, \dot{\lambda}) - \phi(\dot{\varepsilon}, \dot{\lambda}) \geqslant \langle \dot{\varepsilon}^* - \dot{\varepsilon}, \gamma^*\rangle\} \tag{36}$$

The sub-differential $\partial_\sigma \phi^*(\dot{\sigma}, \dot{\lambda}^*)$ may be defined analogously.

Since $\phi$ is differentiable in $\dot{\varepsilon} \in W$, then $\dot{\sigma}$ is uniquely determined by $\partial_\varepsilon \phi(\dot{\varepsilon}, \dot{\lambda}^*)$. Hence,

$$\dot{\sigma} = \nabla_\varepsilon \phi(\dot{\varepsilon}, \dot{\lambda}^*) = E\dot{\varepsilon} - \dot{\lambda} f_\varepsilon = \dot{\sigma}^{\mathrm{e}} - \dot{\sigma}^{\mathrm{d}} \tag{37}$$

Therefore, the following duality relation is valid:

$$\dot{\sigma} = \nabla_{\varepsilon}\phi(\dot{\varepsilon}, \dot{\lambda}^*) \iff \dot{\varepsilon} \in \partial_{\sigma}\phi^*(\dot{\sigma}, \dot{\lambda}) \tag{38}$$

The relation $\dot{\varepsilon} = \nabla_{\sigma}\phi^*(\dot{\sigma}, \dot{\lambda}^*)$ is not verified for all $\dot{\lambda} \in \Lambda_f^+$. In fact, it may exist $\dot{\varepsilon}^1$ and $\dot{\varepsilon}^2$ belonging to $\partial_{\sigma}\phi^*(\dot{\sigma}, \dot{\lambda})$ and correlated to one unique $\dot{\sigma} \in W^*$. Hence, $\phi^*$ is not always differentiable in $\dot{\sigma} \in W^*$.

So far we have been concluded that $\phi$ is a convex, l.s.c., proper and differentiable potential in the variables $\dot{\varepsilon} \in W$ and $\dot{\lambda} \in \Lambda_f^+$. Such potential verifies the optimality conditions found in Eq. (33). Furthermore, there is a conjugate (dual) potential $\phi^*$ related to $\phi$ and verifying Eq. (38).

Taking into account Eq. (38) and the result derived from the Proposition A.3.2, the following relation between the functionals $\phi^*$ and $\phi$ becomes valid:

$$\phi^*(\dot{\sigma}, \dot{\lambda}) + \phi(\dot{\varepsilon}, \dot{\lambda}) = \langle \dot{\sigma}, \dot{\varepsilon} \rangle \iff \dot{\sigma} \in \partial_{\varepsilon}\phi(\dot{\varepsilon}, \dot{\lambda}) \quad \text{and} \quad \dot{\varepsilon} \in \partial_{\sigma}\phi^*(\dot{\sigma}, \dot{\lambda})$$

In what follows, the incremental variational form of the elastic-damage model is presented. It is useful for the goal of numerical simulations.

## 6. Incremental variational form

The restrictive condition $\dot{\lambda} \in \Lambda_f^+$ can be relaxed if one considers an indicator function $I_{\Lambda_f^+}$ defined as:

$$I_{\Lambda_f^+} = \begin{cases} 0 & \text{if } \dot{\lambda} \in \Lambda_f^+ \\ +\infty & \text{if } \dot{\lambda} \in \Lambda_f - \Lambda_f^+ \end{cases} \tag{39}$$

The indicator may be introduced into the model by means of the following asymptotic approximation:

$$I_{\Lambda_f^+} = (-1/\delta)\int_B f\dot{\lambda}\,\mathrm{d}B \quad \text{with } \delta \to 0^+. \tag{40}$$

Therefore, by considering such an approximation the potential defined in the expression (30) becomes:

$$\phi_{\delta}(\dot{\varepsilon}, \dot{\lambda}) = \left\{ 1/2\,\langle E\dot{\varepsilon}, \dot{\varepsilon}\rangle - \langle \dot{\lambda} f_{\varepsilon}, \dot{\varepsilon}\rangle + 1/2\,\langle G\dot{\lambda}, \dot{\lambda}\rangle + \left\langle \frac{f}{\delta}, \dot{\lambda} \right\rangle \right\} \tag{41}$$

$\forall \dot{\lambda} \in \Lambda_f$ and $\delta \to 0^+$. As $\delta \to 0^+$, then $\dot{\sigma} \in \partial_{\varepsilon}\phi_{\delta}(\dot{\varepsilon}, \dot{\lambda})$ converges to $\dot{\sigma} \in \partial_{\varepsilon}\phi(\dot{\varepsilon}, \dot{\lambda})$.

Finally by the substitution of $\dot{\lambda} = \psi\dot{\alpha}$, defined in Eq. (16), one arrives to the equivalent potential:

$$\phi_{\delta}(\dot{\varepsilon}, \dot{\alpha}) = \left\{ 1/2\,\langle E\dot{\varepsilon}, \dot{\varepsilon}\rangle - \langle \psi\dot{\alpha} f_{\varepsilon}, \dot{\varepsilon}\rangle + 1/2\,\langle G\psi\dot{\alpha}, \psi\dot{\alpha}\rangle + \left\langle \frac{f}{\delta}, \psi\dot{\alpha} \right\rangle \right\} \tag{42}$$

$\forall \dot{\alpha} \in \Lambda_g$ with $\delta \to 0^+$. It should be noted that $\dot{\lambda} \in \Lambda_f$ implies $\dot{\alpha} \in \Lambda_g$.

An incremental variational form results from a time discretization as,

$$\Delta\sigma = \dot{\sigma}\Delta t; \quad \Delta\varepsilon = \dot{\varepsilon}\Delta t; \quad \Delta\alpha = \dot{\alpha}\Delta t \tag{43}$$

By substitution of Eq. (43) into Eq. (42), with $\Delta\alpha \in \Lambda_g$, the following potential results as a function of incremental variables:

$$\phi_{\delta}(\Delta\varepsilon, \Delta\alpha) = \left\{ 1/2\,\langle E\Delta\varepsilon, \Delta\varepsilon\rangle - \langle \psi\Delta\alpha f_{\varepsilon}, \Delta\varepsilon\rangle + 1/2\,\langle G\psi\Delta\alpha, \psi\Delta\alpha\rangle + \left\langle \Delta t\frac{f}{\delta}, \psi\Delta\alpha \right\rangle \right\} \tag{44}$$

In particular, taking $\delta = \Delta t$, an extended functional result:

$$\tilde{\phi}(\Delta\varepsilon, \Delta\alpha) = \{ 1/2\,\langle E\Delta\varepsilon, \Delta\varepsilon\rangle - \psi\Delta\alpha f_{\varepsilon}, \Delta\varepsilon\rangle + 1/2\,\langle G\psi\Delta\alpha, \psi\Delta\alpha\rangle + \langle f, \psi\Delta\alpha\rangle \} \tag{45}$$

Locally, with $\Delta\lambda \in \Lambda_f$ or $\forall\Delta\alpha \in \Lambda_g$ the optimality conditions are:

$$[f + f_\varepsilon \cdot \Delta\varepsilon - G\Delta\lambda] = [f + f_\varepsilon \cdot \Delta\varepsilon - \psi G\Delta\alpha] \leqslant 0 \tag{46}$$

$$[f + f_\varepsilon \cdot \Delta\varepsilon - G\Delta\lambda] \cdot \Delta\lambda = [f + f_\varepsilon \cdot \Delta\varepsilon - \psi G\Delta\alpha] \cdot \psi\Delta\alpha = 0 \tag{47}$$

The local Eqs. (46) and (47) are equivalent to a linear complementarity problem which can be solved by mathematical programming methods. In particular, if $f$ is piecewise linear, an algorithm able to solve Eq. (46) gives exact increments $\Delta\lambda$ or $\Delta\alpha$, verifying $f = 0$ at the step $t + \Delta t$. As a consequence, the constitutive relation is represented in an exact way for any $\Delta t$ that does not imply damage followed by unloading.

## 7. Numerical example

The formulation discussed so far is general in the sense that it may be applied to handle with large problems. Nevertheless, we limit here the application to a very simple example. The aim is to point out the possibility given by the solver potential of an exact representation of the constitutive model when a finite step analysis is performed. The characteristics of the structure and constitutive model are illustrated in Fig. 1.

In the uniaxial case, the general relations including the optimality conditions for incremental analysis are:

$$\alpha - (w - \overline{w}) \leqslant 0 \tag{48}$$

$$[-\alpha - (w - \overline{w})]\alpha = 0, \quad \alpha \geqslant 0 \tag{49}$$

$$w = \frac{\bar{\sigma}\bar{\varepsilon}(\varepsilon^* - \varepsilon^e)}{2(\bar{\varepsilon} - \varepsilon^e)} \tag{50}$$

$$f = \varepsilon - \varepsilon^* = \varepsilon - \left[\left(\frac{2w(\bar{\varepsilon} - \varepsilon^e)}{\bar{\sigma}\bar{\varepsilon}}\right) + \varepsilon^e\right] \tag{51}$$

$$f_\varepsilon = 1; \quad f_w = -\frac{2(\bar{\varepsilon} - \varepsilon^e)}{\bar{\sigma}\bar{\varepsilon}} \tag{52}$$

$$E^* = \left(1 - \frac{2w}{E\varepsilon^e\varepsilon^*}\right)E \tag{53}$$

$$\psi = \frac{(f_\varepsilon \otimes \varepsilon^*) \cdot E_{w^d}}{\|f_\varepsilon\|^2}, \quad \psi \leqslant 0 \tag{54}$$

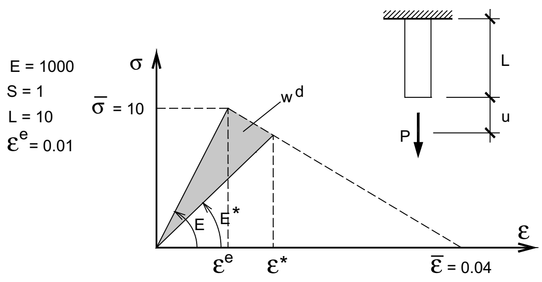


Fig. 1. Truss element submitted to uniaxial traction.

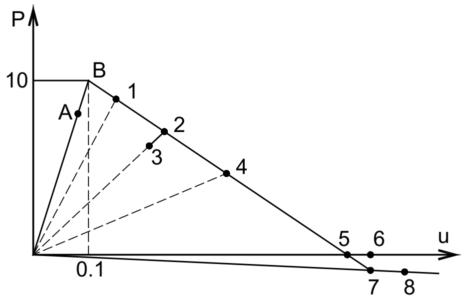


Fig. 2. Numerical response.

$$[f + f_{\varepsilon} \cdot \Delta\varepsilon - f_{w^{d}}\Delta\alpha] \cdot \psi\Delta\alpha = 0, \quad \forall \Delta\alpha \in \Lambda_{g} \tag{55}$$

$$[f + f_{\varepsilon} \cdot \Delta\varepsilon - f_{w^{d}}\Delta\alpha] \leqslant 0, \tag{56}$$

$$\Delta w = -\Delta\alpha \tag{57}$$

$$\Delta\sigma = E^{*}\Delta\varepsilon - \psi\Delta\alpha \tag{58}$$

For this example in particular, it is possible directly to write expressions for the increments of the displacement $\Delta u$ and the load $\Delta P$:

$$[u - u^{*} + \Delta u - Lf_{w^{d}}\Delta\alpha] \leqslant 0 \tag{59}$$

where

$$f = u - u^{*} \tag{60}$$

$$\Delta P = E^{*}\frac{\Delta u}{L} - \psi\Delta\alpha \tag{61}$$

It must be noted that for this case it was adopted $\overline{w} = 0.2$. Therefore,

$$\psi = \frac{-2\varepsilon^{*}}{(0.15w^{d} + 0.01)^{2}} \tag{62}$$

The numerical response obtained is illustrated in Fig. 2.

The history of loading illustrated includes an unloading between the steps 2 and 4, and two possibilities of loading from step 4. In the first possibility the increment of displacement leads to a deformation which is in exact correspondence with the prescribed limit value for the maximum dissipated energy $\overline{w}$. Beyond this point the rigidity is identically annulled for any additional increment. In the second possibility of loading the increment of displacement overpasses the energy limit. As a consequence a residual rigidity appears which is illustrated in Fig. 2 by a dotted line through points 7 and 8. The main numerical results are outlined in Table 1.

### 7.1. On a correction for the strain step

The definition proposed for the slack variable ($\alpha \geqslant 0$) does not eliminate a possibility where:

$$\alpha = 0 \quad \text{and} \quad w - \overline{w} > 0 \tag{63}$$

Table 1
Numerical results

| Points | $\Delta u$ | $\Delta\alpha$ | $\psi$ | $\Delta P$ | $u^*$ | $\varepsilon^*$ | $w$ | $E^*$ | $f$ |
|---|---|---|---|---|---|---|---|---|---|
| A | 0.08 | 0 | 0 | 8 | 0.1 | 0.01 | 0 | 1000 | −0.02 |
| 1 | 0.05 | −0.02 | −200 | 1 | 0.13 | 0.013 | 0.02 | 692.30 | 0 |
| 2 | 0.09 | −0.06 | −153.85 | −3 | 0.22 | 0.022 | 0.08 | 272.73 | 0 |
| 3 | −0.03 | 0 | * | −0.82 | 0.19 | 0.019 | 0.08 | 272.73 | −0.03 |
| 4 | 0.09 | −0.04 | −90.91 | −1.182 | 0.28 | 0.028 | 0.12 | 142.85 | 0 |
| 5 | 0.12 | −0.08 | −71.43 | −4 | 0.4 | 0.04 | 0.2 | 0 | 0 |
| 6 | 0.03 | 0 | −71.43 | 0 | 0.43 | 0.043 | 0.2 | 0 | 0 |
| 7 | 0.15 | −0.1 | −71.43 | −5 | 0.43 | 0.043 | 0.22 | −23.25 | 0 |
| 8 | 0.03 | 0 | −46.51 | −0.07 | 0.46 | 0.046 | 0.22 | −23.25 | 0 |

Such a case implies a violation of the following condition:

$$\varepsilon \leqslant \bar{\varepsilon} \tag{64}$$

and a residual rigidity can appear. In order to avoid that inconvenience a relaxation function is introduced, being defined as:

$$j(\beta, \varepsilon) = -\beta - (\bar{\varepsilon} - \varepsilon) \leqslant 0 \quad \text{with } \beta \geqslant 0 \tag{65}$$

Therefore, a complementary problem results:

$$\begin{cases} \text{if } j < 0 \Rightarrow \beta = 0 \quad \text{and} \quad \varepsilon < \bar{\varepsilon} \\ \text{if } j = 0 \Rightarrow \varepsilon \geqslant \bar{\varepsilon} \Rightarrow \beta = \varepsilon - \bar{\varepsilon} \geqslant 0 \end{cases} \tag{66}$$

where the condition $j < 0$ and $\beta = 0$ implies that Eq. (64) is satisfied, while $j = 0$ and $\beta \geqslant 0$ represents the violation of it. It is now possible to set a complementary relation between $\alpha$ and $\beta$ as:

$$\alpha \cdot \beta = 0 \iff \begin{cases} \alpha = \overline{w}^{\mathrm{d}} - w^{\mathrm{d}} \geqslant 0;\ \beta = 0 \text{ and } \varepsilon \geqslant \bar{\varepsilon} \\ \alpha = 0;\ \overline{w}^{\mathrm{d}} \leqslant w^{\mathrm{d}} \text{ and } \beta = \varepsilon - \bar{\varepsilon} \geqslant 0 \end{cases} \tag{67}$$

The first possibility ($\alpha \geqslant 0$ and $\beta = 0$) implies that $g \cdot \alpha = 0$, with $g \leqslant 0$, and that Eq. (64) is satisfied. The second possibility ($\alpha = 0$ and $\beta > 0$) implies violation of $g \leqslant 0$ and Eq. (64). Finally, a correction for the total strain $\epsilon$ in the current step results from:

$$\varepsilon = \varepsilon^* \quad \text{if } \alpha \geqslant 0 \text{ and } \beta = 0; \qquad \varepsilon = \varepsilon^* - \beta \quad \text{if } \alpha = 0 \text{ and } \beta > 0 \tag{68}$$

where $\epsilon^*$ is the total strain presently imposed. With the corrected value of $\epsilon$ it is then possible to determine the damage energy variable $w$ and the elastic-damage tensor $E$.

Coming back to the numerical example, by applying the correction given by Eq. (68) and considering the results illustrated in Fig. 3, it may be observed that:

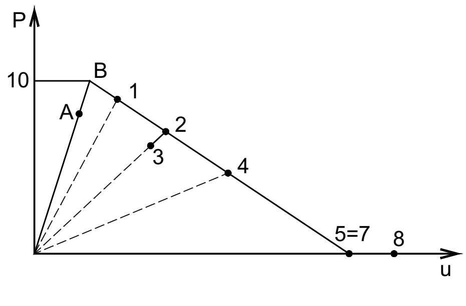


Fig. 3. Corrected numerical response.

(i) the points A, 1, 2, 3 and 4 present $\alpha \geqslant 0$ and $\beta = 0$ or $\varepsilon = \varepsilon^*$;
(ii) imposing a displacement $\Delta u = 0.15$ from the point 4 the correction procedure leads to $\beta = \varepsilon - \bar{\varepsilon} = 0.003$ and $\varepsilon = \varepsilon^* - \beta = 0.043 - 0.003 = 0.04$. By substitution of such value into Eqs. (50) and (53) one can obtain $w = 0.2$ and $E^* = 0$ (point 7).
(iii) beyond that point, any additional displacement (point 8) gives to $\alpha = 0$, $w = 0.2$ and $E^* = 0$.

## 8. Extension to the non-associative case

The potentials $\phi$, $\phi_\delta$ and $\tilde{\phi}$ defined by (30), (42) and (45), respectively, may be defined in order to include the non-associative case. In fact, the relations of the associative case may be extended straightforward to the non-associative case. This is done by using the vector $h$, which is defined in relation (4) such that $\dot{\sigma} = -\dot{\lambda} h = \psi \dot{\alpha} h$.

## 9. Conclusions

A convex damage potential, written as the sum of a potential of the strains and a potential of the damage variable has been defined. Applying convexity concepts and assuming some properties, the existence of a convex conjugate potential was proved. Also, it was shown that with the sub-differential sets of that potential it is possible to derive the constitutive relation for an elastic-damage material in rates, including the complementarity and consistency conditions. The incremental form of the model was then introduced aiming the numerical simulations. A simple example was proposed to point out the possibility of an exact verification of the constitutive model if linear softening is assumed and the displacement increment does not violate the adopted limit for the total dissipated energy. If this condition for the displacement increment does not hold a sufficient accurate response can be obtained providing small steps or by using a step deformation correction. It should also be pointed out that the solver proposition here stated is general in the sense that it may be extended to handle with more general constitutive relations. For instance, both elastoplastic-damage response and different softening laws could be considered. Finally, the formulation here proposed is feasible to establish kinematical, equilibrium and mixed principles in the solid mechanics.

## Appendix A

This Appendix A is divided in three parts and it is in agreement with the results found in Ekeland and Temam (1976) and Rockafellar (1970).

### *A.1. Proper functional, coercive functional and semi-continuity of a functional*

The definitions and propositions included here were used to prove the results proposed in Section 4.

The extrema values of a semi-continuous real functional in a minimisation problem may be $-\infty$ or $+\infty$. Obviously, the results depend of the assumptions done over the functionals and of the sets on which they operate, as will be seen in this Appendix A.

**Definition A.1.1** (*Extended real set*). The extended real set $\overline{\Re}$ is defined as the union of the real set $\Re$ including the limits values at the $-\infty$ and $+\infty$, i.e. $\overline{\Re} = [-\infty, +\infty]$.

**Definition A.1.2** (*Proper functional*). Let $X$ be a normalised vectorial space (NVS). Then a real functional $f : X \to \overline{\Re}$ is *proper* if $f$ is not identically equal to $+\infty$ and if $f$ does not reach the $-\infty$ value.

**Definition A.1.3** (*Semi-continuity of a functional*). A functional $f : X \to \overline{\Re}$ is said to be l.s.c. at $x_0 \in X$ if for all sequence $\{x_n\}$, $x_n \in X$, convergent to $x_0 (x_n \to x_0)$, the following condition is verified

$$\lim_{n\to\infty} \inf \ f(x_n) \geqslant f(x_0);$$

**Definition A.1.4** (*Weak semi-continuity*). The functional $f : X \to \overline{\Re}$ is said to be weakly l.s.c. at $x_0 \in X$ if for all sequence $\{x_n\}$, $x_n \in X$, weakly convergent to $x_0$, it is verified that:

$$\lim_{n\to\infty} \inf \ f(x_n) \geqslant f(x_0).$$

**Definition A.1.5** (*Coercive functional*). The functional $f : X \to \overline{\Re}$ is said coercive if for all divergent sequence $\{x_n\}$, $x_n \in X$, such that $\|x_n\|_E \to +\infty$, ($\|\cdot\|_E$ is the Euclidean norm) then $f(x_n) \to +\infty$, i.e.:

$$\lim_{\|x_n\|\to\infty} f(x_n) = +\infty$$

**Definition A.1.6** (*Growing property*). The functional $f : X \to \overline{\Re}$ is said to present the Growing property at $x_0 \in X$ if there is an scalar $r > 0$ such that $f(x) \geqslant f(x_0)$, $\forall x \in X$, which satisfy:

$$\|x - x_0\|_E > r.$$

**Proposition A.1.1.** *If $f : X \to \overline{\Re}$ is coercive, then $f$ presents the Growth Property at all $x \in X$.*

*A.2. The existence of a minimum for a continuum convex functional*

The propositions that follow were used in Section 4.

**Proposition A.2.1.** *Let $X$ be a NVS and $C \subset X$ a non-empty convex closed subset.*

*If $f : X \to \overline{\Re}$, is a continuum convex and coercive functional, then $f$ is weakly l.s.c. and presents the growing property at some $x_0 \in C$.*

**Proposition A.2.2.** *Let $X$ be a NVS and $C \subset X$ a non-empty convex closed subset.*

*If $f : X \to \overline{\Re}$, is a weakly l.s.c. functional presenting the growth property at some $x_0 \in C$, then $f$ is bounded, reaching its minimum in $C$.*

*A.3. Conjugate convex functional*

The propositions that follow were used in Section 5.

**Definition A.3.1** (*Fenchel's conjugated functional or the Legendre's transform*). Let $X$ be a NVS, $C \subset X$ a non-empty convex closed subset of it and $X^*$ the dual NVS of $X$.

Being $f$ a continuum convex functional defined on $C$, it is possible to define a conjugated set, denoted by $C^*$ and expressed as:

$$C^* = \left\{ x^* \in X^* : \sup_{x \in C} [\langle x^*, x \rangle - f(x)] < \infty \right\},$$

Furthermore, there is a conjugated functional, related to $f$ and defined on $x^* \in C^*$. Such a functional is denoted by $f^*$, and given by:

$$f^*(x^*) = \sup_{x \in C} [\langle x^*, x \rangle - f(x)].$$

$f^*$ is known as the Fenchel's conjugate functional or the Legendre's transform.

**Proposition A.3.1.** *If the set $[f, C]$ is a non-empty convex closed set then, the conjugate set $C^*$ and the conjugate functional $f^*$ are convex and $[f^*, C^*]$ is a non-empty convex closed subset of $\Re \times X^*$. In such case: $[f^*, C^*] = [f, C]^*$.*

**Proposition A.3.2.** *Consider $C$ and $C^*$, $f$ and $f^*$, verifying the assumptions of Proposition* A.3.1*, then the following inequality is valid:*

$$f^*(x^*) + f(x) \geqslant \langle x, x^* \rangle, \quad \forall x^* \in C^* \quad \text{and} \quad \forall x \in C.$$

*Furthermore, $x$ and $x^*$ are said conjugates of $f$ and $f^*$ if the equality is verified:*

$$f^*(x^*) + f(x) = \langle x, x^* \rangle,$$

*or, in the equivalent form:*

$$x^* \in \partial f(x) \quad \text{and} \quad x \in \partial f^*(x^*).$$

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